

# The CIA (consistency in aggregation) approach

## A new economic approach to elementary indices

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\* This presentation represents the author's personal opinions and does not necessarily reflect the views of the Deutsche Bundesbank or its staff.

# Outline

1. Motivation
2. Background
3. Economic approach
4. Consistency approach
5. Empirical results
6. Discussion

*“Elementary, my dear Watson!” (Sherlock Holmes)*

## 1. Motivation

### Two-staged index calculation

- Practical consumer price indices are constructed in two stages:
  1. a first stage at the lowest level of aggregation where price information is available but associated expenditure or quantity information is not available and
  2. a second stage of aggregation where expenditure information is available at a higher level of aggregation.
- Although the scope of the discussion is on the choice of the index formula at the elementary level, this choice eventually depends on the target price index at the aggregate level.
  - The importance of the elementary index cannot be emphasised enough; biases at this level are more severe than the pros and cons of the formula at the aggregate level.
- The consumer price index (CPI) is not intended to measure the cost of living (COLI), rather, it is a cost of goods index (COGI).

# 1. Motivation

## Bilateral price indices

- We specify **two accounting periods**,  $t \in \{0, 1\}$ , for which we have micro price and quantity data for  $n$  commodities (bilateral index context).
- Denote the **price and quantity** of commodity  $i \in \{1, \dots, n\}$  in period  $t$  by  $p_i^t$  and  $q_i^t$ , respectively.
- A very simple approach to the determination of a price index over a group of commodities is **the (fixed) basket approach**.
- Define the **Lowe (1823) price index**,  $P_{Lo}$ , as follows:
$$- P_{Lo} = \frac{\sum_{i=1}^n p_i^1 \cdot q_i}{\sum_{i=1}^n p_i^0 \cdot q_i}.$$
- There are **two natural choices** for the reference basket:
  - the period 0 commodity vector  $\mathbf{q}^0 = (q_1^0, \dots, q_n^0)$  or
  - the period 1 commodity vector  $\mathbf{q}^1 = (q_1^1, \dots, q_n^1)$ .

# 1. Motivation

## Laspeyres, Paasche and Fisher

– These two choices lead to

- the **Laspeyres (1871) price index**  $P_L$ , if we choose  $q = q^0$ , and
- the **Paasche (1874) price index**  $P_P$ , if we choose  $q = q^1$ :

$$- P_L = \frac{\sum_{i=1}^n p_i^1 \cdot q_i^0}{\sum_{i=1}^n p_i^0 \cdot q_i^0}, \quad P_P = \frac{\sum_{i=1}^n p_i^1 \cdot q_i^1}{\sum_{i=1}^n p_i^0 \cdot q_i^1}.$$

– According to the CPI Manual (ILO et al., 2004), “**the Paasche and Laspeyres price indices are equally plausible.**”

– Taking an evenly weighted average of these basket price indices leads to **symmetric averages**.

– **The geometric mean**, which leads to the **Fisher (1922) price index**,  $P_F$ , is defined as:

$$- P_F = \sqrt{P_L \cdot P_P} .$$

## 1. Motivation Test approach

- Looking at the mathematical properties of index number formulae leads to the test or axiomatic approach to index number theory.
- In this approach, desirable properties for an index number formula are proposed, and it is then attempted to determine whether any formula is consistent with these properties or tests.
- It is worth mentioning that the time reversal test, which is the main justification of the Fisher, Walsh and Törnqvist price indices, is meaningful only in interspatial comparisons (then as the country reversal test).
- In intertemporal comparisons, however, the direction of comparison is not arbitrary (it is not unjustified to prefer a forward movement to moving backwards) (cf. von der Lippe, 2007).
- Hence, the analogy borrowed from the literature on international purchasing power parity comparisons has had to be discarded on further reflection since it is a straw man argument.

## 2. Background Elementary indices

- Suppose that there are  **$M$  lowest-level items or specific commodities** in a chosen elementary category.
- Denote the period  $t$  price of item  $m$  by  $p_m^t$  for  $t \in \{0, 1\}$  and for items  $m \in \{1, \dots, M\}$ .
- The **Dutot (1738) elementary price index**,  $P_D$ , is equal to the *arithmetic* average of the  $M$  period 1 prices divided by the *arithmetic* average of the  $M$  period 0 prices.
- The **Carli (1764) elementary price index**,  $P_C$ , is equal to the *arithmetic* average of the  $M$  item price ratios or price relatives,  $p_m^1/p_m^0$ .
- The **Jevons (1865) elementary price index**,  $P_J$ , is equal to the *geometric* average of the  $M$  item price ratios or price relatives,  $p_m^1/p_m^0$ , or the *geometric* average of the  $M$  period 1 prices divided by the *geometric* average of the  $M$  period 0 prices.

$$- P_D = \frac{\frac{1}{M} \sum_{m=1}^M p_m^1}{\frac{1}{M} \sum_{m=1}^M p_m^0}, P_C = \frac{1}{M} \sum_{m=1}^M \frac{p_m^1}{p_m^0}, P_J = \sqrt[M]{\prod_{m=1}^M \frac{p_m^1}{p_m^0}} = \frac{\sqrt[M]{\prod_{m=1}^M p_m^1}}{\sqrt[M]{\prod_{m=1}^M p_m^0}}$$

## 2. Background

### Is the Carli index really “upward biased”?

- The sole argument frequently put forward why the Carli index should be abandoned, is **the claim that it has an “upward bias”** with reference to the time reversal test (cf. Diewert, 2012):
- $P_C(p^0, p^1) \cdot P_C(p^1, p^2) = P_C(p^0, p^1) \cdot P_C(p^1, p^0) \geq 1 = P_C(p^0, p^0)$  for  $p^2 = p^0$ .
- But **this argument is useless in the bilateral index context** where we can compare the two periods under consideration directly, i.e. there is no bias at all:
- $P_C(p^0, p^2) = P_C(p^0, p^0) = 1$  for  $p^2 = p^0$ .
- In the context of chain indices, **the elementary aggregates only feed into the higher-level indices** in which the elementary price indices – comparing periods  $t-1$  and  $t$  (!) – are averaged using a set of pre-determined weights (chain indices are non-aggregable); **the Dutot, Carli and Jevons indices are, thus, not chain-linked**.
- What is more, it apparently fell into oblivion that **the then chain-linked Laspeyres, Paasche, Fisher, Walsh and Törnqvist price indices are subject to chain drift**; i.e. all chain indices are path dependent, which is the opposite of transitivity.

## 2. Background

### Why the Carli index really should be abandoned!

- There might be **good reasons in practice for relying on geometric averaging**, i.e. the Jevons index, **even where arithmetic averaging**, i.e. the Carli index, **would be theoretically superior**.
  - Notwithstanding that **the Carli formula has no bias with respect to the desired aggregate index**, it can show large fluctuations in the presence of outliers and *in the context of chain indices when weights are not available*:
  - $\text{MSE}(P_J) = \text{Var}(P_J) + \text{Bias}^2(P_J) < \text{Var}(P_C) = \text{MSE}(P_C) \Leftrightarrow \text{Var}(P_J) \ll \text{Var}(P_C)$ .
  - **The Jevons formula**, on the other hand, **is more robust** in these circumstances, which is due to the fact that **it satisfies the circularity test** (however, it is noteworthy that **when item substitution takes place none of the elementary index formulae will meet the circularity test**):
    - $P_C(p^0, p^2) - P_C(p^0, p^1) \cdot P_C(p^1, p^2) = \text{Cov}[p^1/p^0, p^2/p^1] < 0$  for  $p^2 = p^0$ .
- Still, **the best solution of course would be disaggregation**, i.e. implementing low-level weights in order to **make the now-lowest level more homogeneous**; or, to paraphrase Galileo Galilei: “Weight what can be weighted and make weightable what is not so.”

### 3. Economic approach

- The CPI Manual, paragraphs 20.71-20.86, has a section in it which describes **an economic approach to elementary indices**.
- This section has sometimes been used to **justify the use of the Jevons index**, i.e. the geometric mean, **over the use of the Carli index**, i.e. the arithmetic mean, or vice versa **depending on how much substitutability exists** between items within an elementary stratum.
- **This is a misinterpretation of the analysis** that is presented in this section of the Manual.
- **Thus, the economic approach cannot be applied at the elementary level unless price and quantity information are both available.**
- **Such information is typically not available**, which is exactly the reason elementary indices are used rather than target indices. (Diewert, 2012, “Consumer Price Statistics in the UK”)

## 4. Consistency approach

### Consistency in aggregation

- The **consistency in aggregation (CIA) approach** newly developed (Mehrhoff, 2010, Jahr. Nationalökon. Statist.) fills the void of **guiding the choice of the elementary index** (for which weights are not available) **that corresponds to the characteristics of the index at the second stage** (where weights are actually available).
- It contributes to the literature by looking at how **numerical equivalence between an unweighted elementary index and a weighted aggregate index** can be achieved, **independent of the axiomatic properties**.
- *Consistency in aggregation* means that **if an index is calculated stepwise by aggregating lower-level indices** to obtain indices at progressively higher levels of aggregation, **the same overall result** should be obtained **as if the calculation had been made in one step**.

## 4. Consistency approach

### Elementary index bias

- Thus, a relevant, although often neglected, issue in practice is **the numerical relationship between elementary and aggregate indices**.
- This is because if the elementary indices do not reflect the characteristics of the aggregate index, **a two-staged index can lead to a different conclusion** than that reached by the price index calculated directly from the price relatives.
- An elementary index in the CPI is **biased if its expectation differs from its measurement objective**.
- This elementary index bias is **applicable irrespective of which unweighted index is used**.
- **In other words, if the elementary index coincides (in expectation) with the aggregate index, the bias will vanish.**

## 4. Consistency approach

### A thought experiment

- In this workshop, **we are only concerned with the elementary index formula that is consistent in aggregation with the – given – desired aggregate index**, e.g. the Laspeyres, Paasche and Fisher price indices.
  - **The choice of the higher-level index itself is out of the scope of this morning's studies** but it is the topic of this afternoon's workshop organised by the Australian Bureau of Statistics.
  - More particularly, **we are not at all dealing with upper-level substitution!**
- **The consumer preferences will**, hence, **not guide the search for the correct formula for the “true” cost of living index at the aggregate level** but rather be employed **to examine the structure of expenditures at the elementary level**, i.e. their distribution among items, allowing the calculation of consistent price indices.
- Let **six commodities** be stratified into **two elementary aggregates** (three each).
  - Assume that the *homogeneous preferences* can be represented by a **Cobb-Douglas function within each stratum**, i.e. **consumers vary the quantities in inverse proportion to the changes in relative prices so that expenditure shares remain constant** (unity elasticity).

## 4. Consistency approach

### Full information on prices and quantities

Elementary aggregate	Commodity	Prices		Quantities	
		Period 0	Period 1	Period 0	Period 1
I	1	1.0	1.2	1.6	1.7
	2	1.0	3.0	1.6	0.7
	3	1.5	1.3	1.1	1.6
II	4	0.5	0.7	7.0	3.8
	5	1.7	1.4	2.1	1.9
	6	0.6	0.8	5.8	3.3
Aggregate index					
		$P_L$	$P_P$	$P_F$	$P_{GL}$
		1.3437	1.1891	1.2640	1.2429

## 4. Consistency approach Limited information on expenditures

Elementary aggregate	Commodity	Prices		Expenditure shares	
		Period 0	Period 1	Period 0	Period 1
I	1	1.0	1.2		
	2	1.0	3.0	0.3141	0.4348
	3	1.5	1.3		
II	4	0.5	0.7		
	5	1.7	1.4	0.6859	0.5652
	6	0.6	0.8		
		Elementary index			
Elementary aggregate		$P_C$	$P_H$	$P_{CSWD}$	$P_J$
I		1.6889	1.2928	1.4776	1.4612
II		1.1856	1.1200	1.1523	1.1541

## 4. Consistency approach

### Two-staged calculation of price indices

Higher-level index	Elementary index				
	$P_C$	$P_H$	$P_{CSWD}$	$P_J$	
$P_L$	1.3437	<u>1.3437</u>	1.1743	1.2545	1.2506
$P_P$	1.1891	1.3621	<u>1.1891</u>	1.2743	1.2702
$P_F$	1.2640	1.3529	1.1817	<u>1.2644</u>	1.2603
$P_{GL}$	1.2429	1.3250	1.1716	1.2460	<u>1.2429</u>

- The two-staged Laspeyres price index with Carli indices at the lower level coincides with the aggregate Laspeyres price index.
- The Jevons index would have a downward bias for the aggregate Laspeyres price index but it is a reasonable approximation for the Fisher price index.
- This clearly shows that no statistical one-size-fits-all approach exists.
- The CIA approach works even without knowing the lower-level quantities!

## 4. Consistency approach

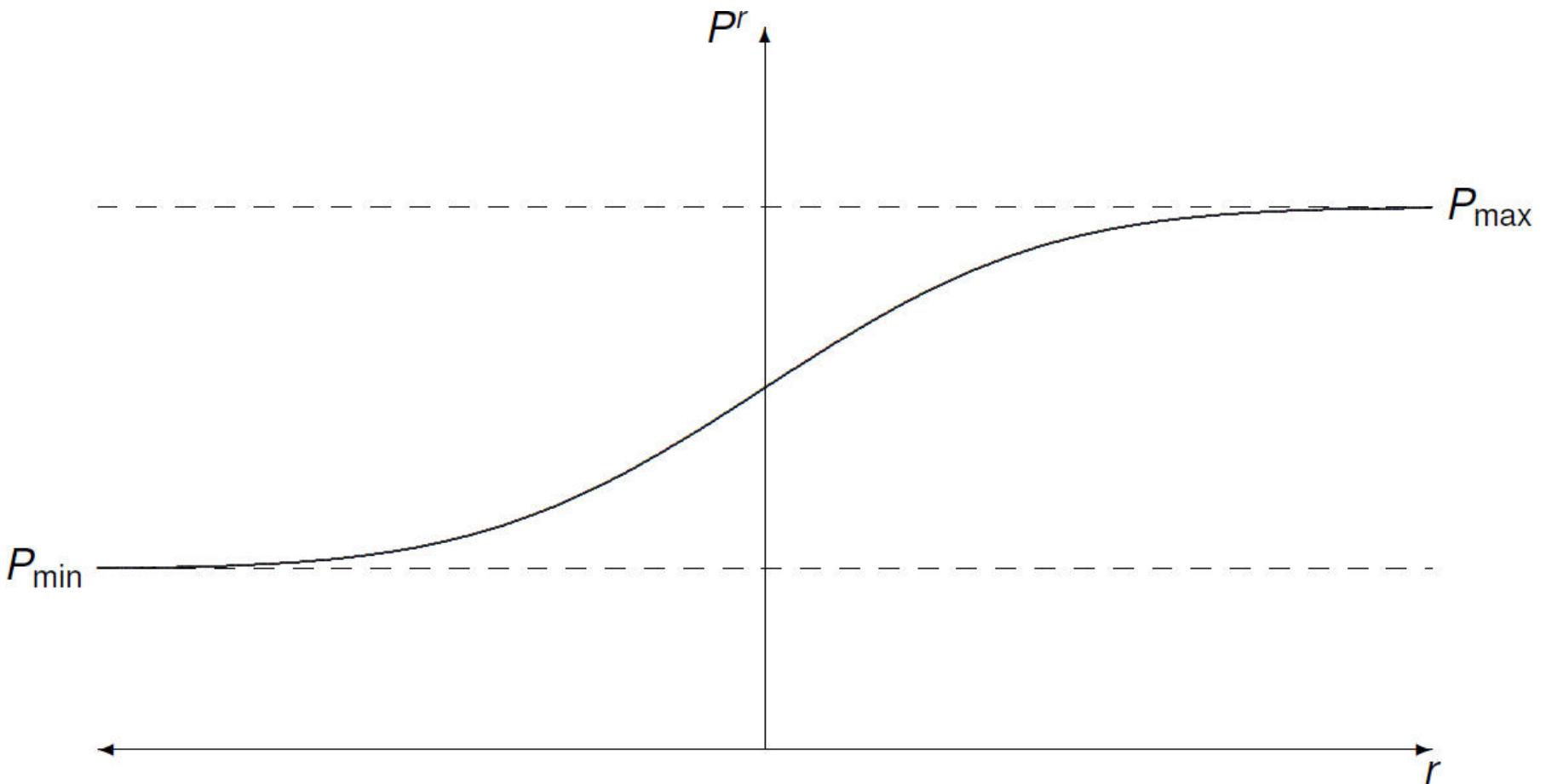
### Generalised means

- A single comprehensive framework, known as *generalised means*, **unifies the aggregate and elementary levels**.
- **The generalised mean** of order  $r$  for the  $M$  item price ratios or price relatives,  $p_m^1/p_m^0$ , is defined as follows:

$$- P^r = \begin{cases} \sqrt[r]{\frac{1}{M} \sum_{m=1}^M \left( \frac{p_m^1}{p_m^0} \right)^r} & \text{if } r \neq 0, \\ \sqrt[M]{\prod_{m=1}^M \frac{p_m^1}{p_m^0}} & \text{if } r = 0. \end{cases}$$

- The generalised mean represents **a whole class of unweighted elementary indices**, such as the Carli and Jevons indices for  $r = 1$  and  $r = 0$ , respectively.

## 4. Consistency approach Typical shape



$$P_{\min} = \min \left( \left\{ \frac{p_m^1}{p_m^0} \right\} \right), P_{\max} = \max \left( \left\{ \frac{p_m^1}{p_m^0} \right\} \right)$$

## 4. Consistency approach Constant elasticity of substitution

- However, **an analytical derivation** of the concrete generalised mean of a weighted aggregate index **is not possible** without further assumptions.
- Hence, both the generalised mean and the target indices are expanded by **a second-order Taylor series approximation** around the point  $\ln p_m^t = \ln p^t$  for all  $m \in \{1, \dots, M\}$ ,  $t \in \{0, 1\}$ .
- Next, it is usually adequate to assume **a constant elasticity of substitution (CES)** approximation in the context of **approximating a consumer's expenditures** on the  $M$  commodities under consideration.
- Finally, it is shown that the choice of the elementary indices which correspond to the desired aggregate ones can be based on **the elasticity of substitution alone**.
- **Thus, a feasible framework is provided which aids the choice of the corresponding elementary index.**

## 4. Consistency approach CES aggregator function

- It is supposed that **the unit cost function** has the following functional form:

$$-c(\mathbf{p}) = \begin{cases} \alpha_0 \cdot \left( \sum_{m=1}^M \alpha_m \cdot p_m^{1-\sigma} \right)^{1/(1-\sigma)} & \text{if } \sigma \neq 1, \\ \alpha_0 \cdot \prod_{m=1}^M p_m^{\alpha_m} & \text{if } \sigma = 1, \end{cases}$$

- where the  $\alpha_m$  are non-negative consumer preference parameters with  $\sum_{m=1}^M \alpha_m = 1$ .
- This unit cost function corresponds to a **CES aggregator or utility function**.
- The parameter  $\sigma$  is **the elasticity of substitution**:
  - When  $\sigma = 0$ , the underlying preferences are **Leontief preferences**.
  - When  $\sigma = 1$ , the corresponding utility function is a **Cobb-Douglas function**.

## 4. Consistency approach Laspeyres and Paasche price indices

- A generalised mean of order  $r$  **equal to the elasticity of substitution ( $\sigma$ )** yields approximately the same result as the **Laspeyres price index**.
- Hence, if the elasticity of substitution is one (Cobb-Douglas preferences), for example,  $r$  must equal one and **the Carli index at the elementary level will correspond to the Laspeyres price index as target index**.
- However, if the **Paasche price index** should be replicated, the order of the generalised mean must **equal minus the elasticity of substitution**, in the above example minus one.
- Thus, **the harmonic index gives the same result** and therefore, in this case it should be used at the elementary level.
- **Only if the elasticity of substitution is zero** (Leontief preferences), **the Jevons (Dutot) index corresponds to both the Laspeyres and Paasche price indices** – which in this case coincide.

## 4. Consistency approach

### Fisher price index

- **The Fisher price index** is derived from the Laspeyres and Paasche price indices as their geometric mean.
- Owing to **the symmetry of the generalised means which correspond to the Laspeyres and Paasche price indices**, a quadratic mean corresponds to the Fisher price index, where  $q$  must equal two times the elasticity of substitution.
- **A quadratic mean** of price relatives of order  $q$  is defined as follows:  
$$- P^q = \sqrt{P^{r=q/2} \cdot P^{r=-q/2}}.$$
- **The index is symmetric**, i.e.  $P^q = P^{-q}$ . Furthermore, **it is either increasing or decreasing** in  $|q|$ , depending on the data.
- Note that **a quadratic mean** of order  $q$ ,  $P^q$ , **should not be mistaken for the quadratic index**,  $P^{r=2}$ .

## 4. Consistency approach

### Quadratic means

- Dalén (1992), and Diewert (1995) show via a Taylor series expansion that **all quadratic means approximate each other** to the second order.
- However, as Hill (2006) demonstrates, the limit of  $P^q$  if  $q$  diverges is  $\sqrt{P_{\min} \cdot P_{\max}}$ ; he concludes that **quadratic means are not necessarily numerically similar**.
- For  $\sigma = 0$  ( $q = 0$ ) **the quadratic mean** becomes **the Jevons index**.
- For  $\sigma = .5$  ( $q = 1$ ) an index results, which was first described by Balk (2005, 2008) as the unweighted Walsh price index and independently devised by Mehrhoff (2007, pp. 45-46) as **a linear approximation to the Jevons (CSWD) index**; hence, this index number formula is referred to as **the Balk-Mehrhoff-Walsh index**, or, for short, “BMW”.
- Lastly, one arrives at **the CSWD index** (Carruthers, Sellwood and Ward, 1980, and Dalén, 1992) for  $\sigma = 1$  ( $q = 2$ ), which is **the geometric mean of the Carli and harmonic indices**.

## 4. Consistency approach Corresponding elementary indices

Laspeyres (generalised mean:  $\sigma$ )

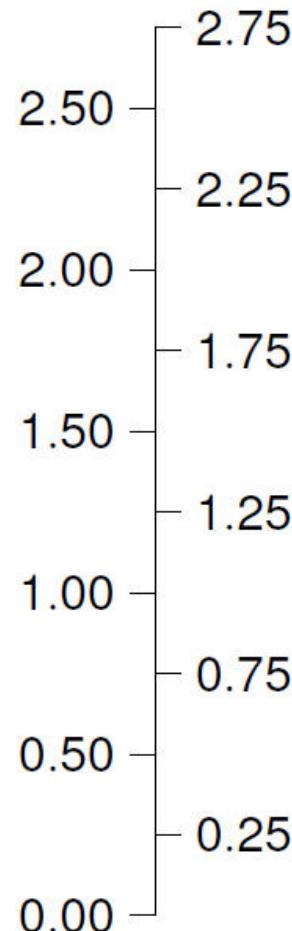
Paasche (generalised mean:  $-\sigma$ )

quadratic  
reciprocal quadratic

Carli  
harmonic

Jevons

$\sigma$



Fisher (quadratic mean:  $2\sigma$ )

quintic

quartic

cubic

CSWD

BMW

Jevons

## 5. Empirical results

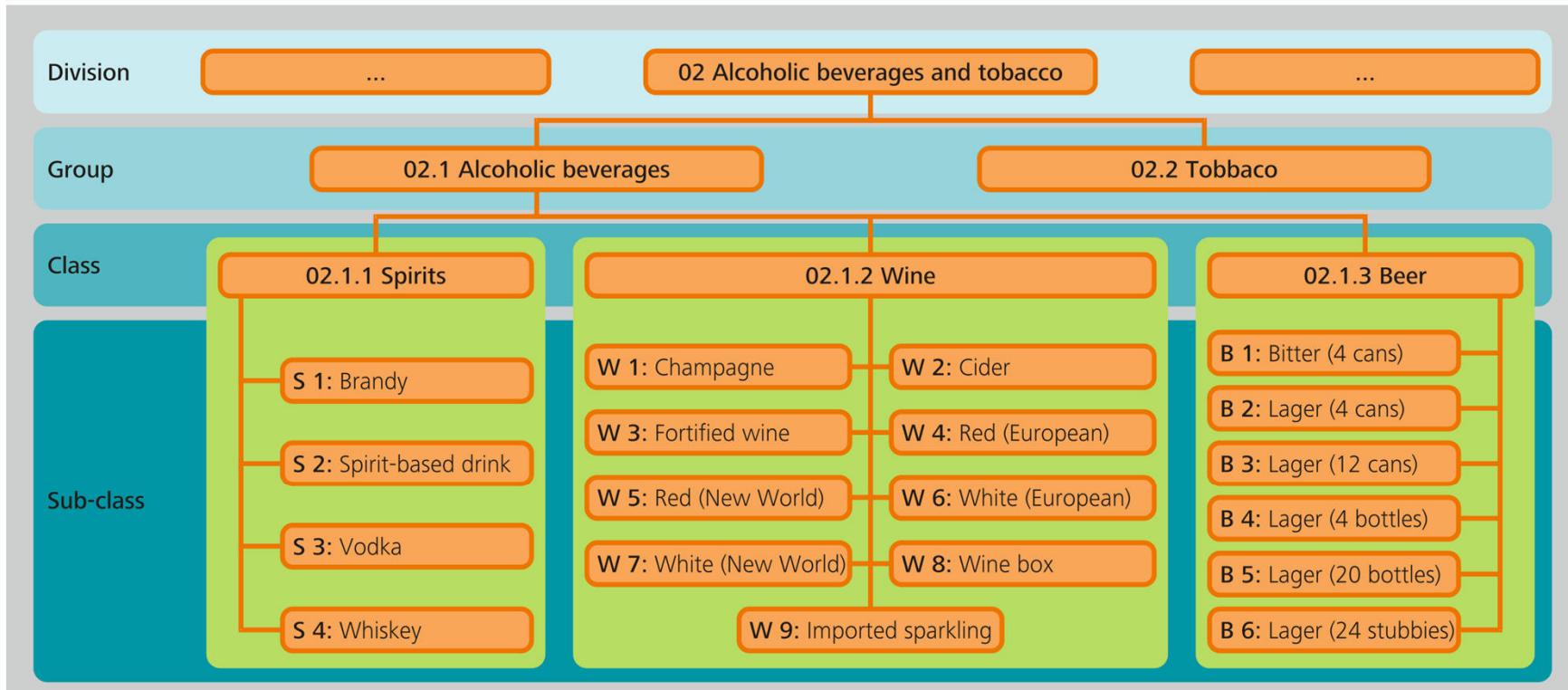
### Alcoholic beverages in the UK

- As an empirical application, **detailed expenditure data** from Kantar Worldpanel for elementary aggregates within the COICOP group of **alcoholic beverages in the UK** are analysed.
- The data cover the period **from January 2003 to December 2011**; the data set consists of transaction level data, which records inter alia purchase price and quantity, and includes **192,948 observations** after outlier identification.
- **The elasticity of substitution** is estimated in the framework of a log-linear model by means of **ordinary least squares**. (Note that the consumer preference parameters are removed via differencing products common to adjacent months and, thus, there is no need for application of seemingly unrelated regression.)
- As a robustness check to the CES model based results, **the generalised mean** which minimises relative bias and root mean squared relative error to the desired aggregate index is found directly by **numerical optimisation techniques**. (Rather than at the aggregate transaction level, like the econometric method, this analysis, however, is performed one level above – at the elementary level.)

# 5. Empirical results

## COICOP structure for alcoholic beverages

### COICOP structure of the overall HICP



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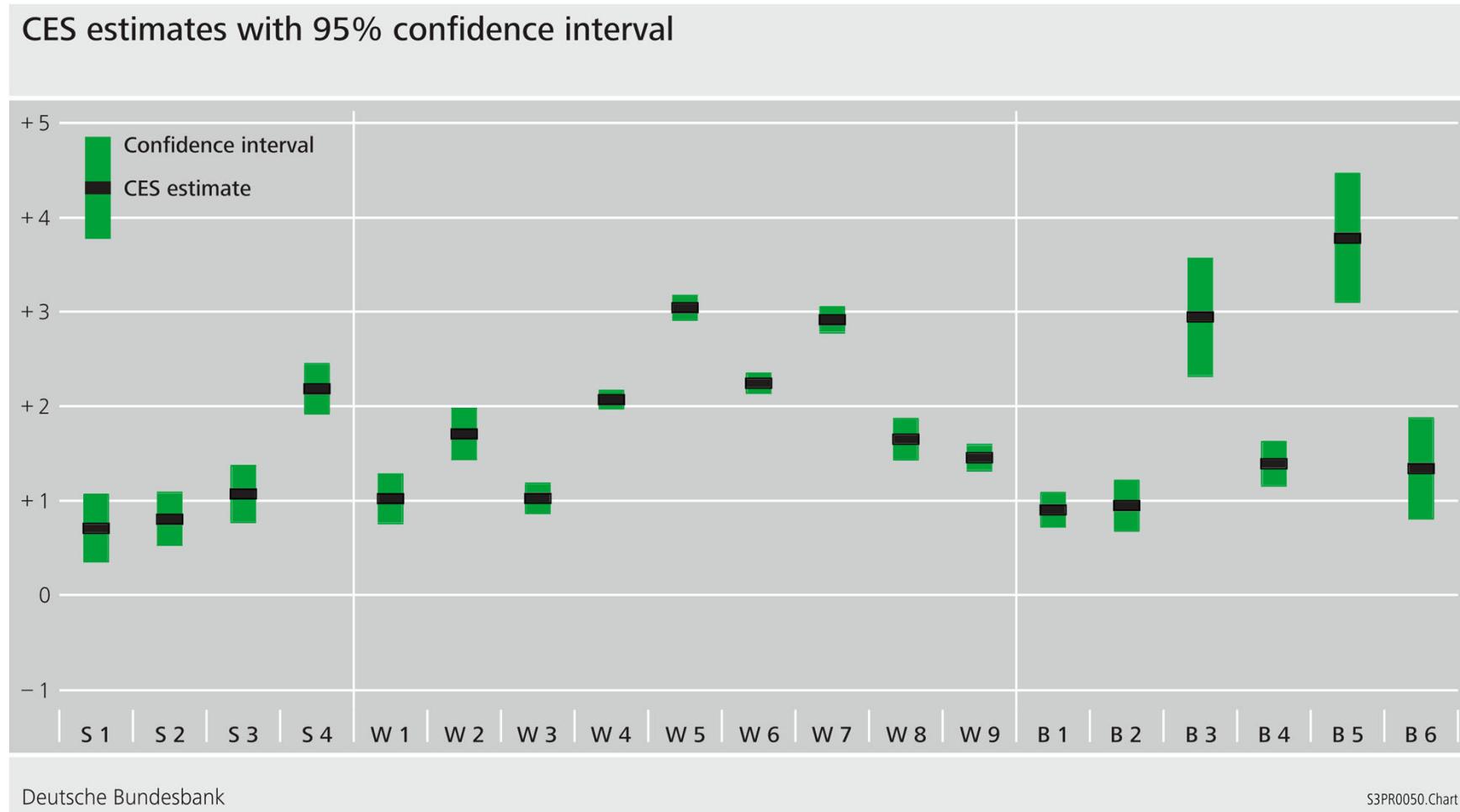
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## 5. Empirical results

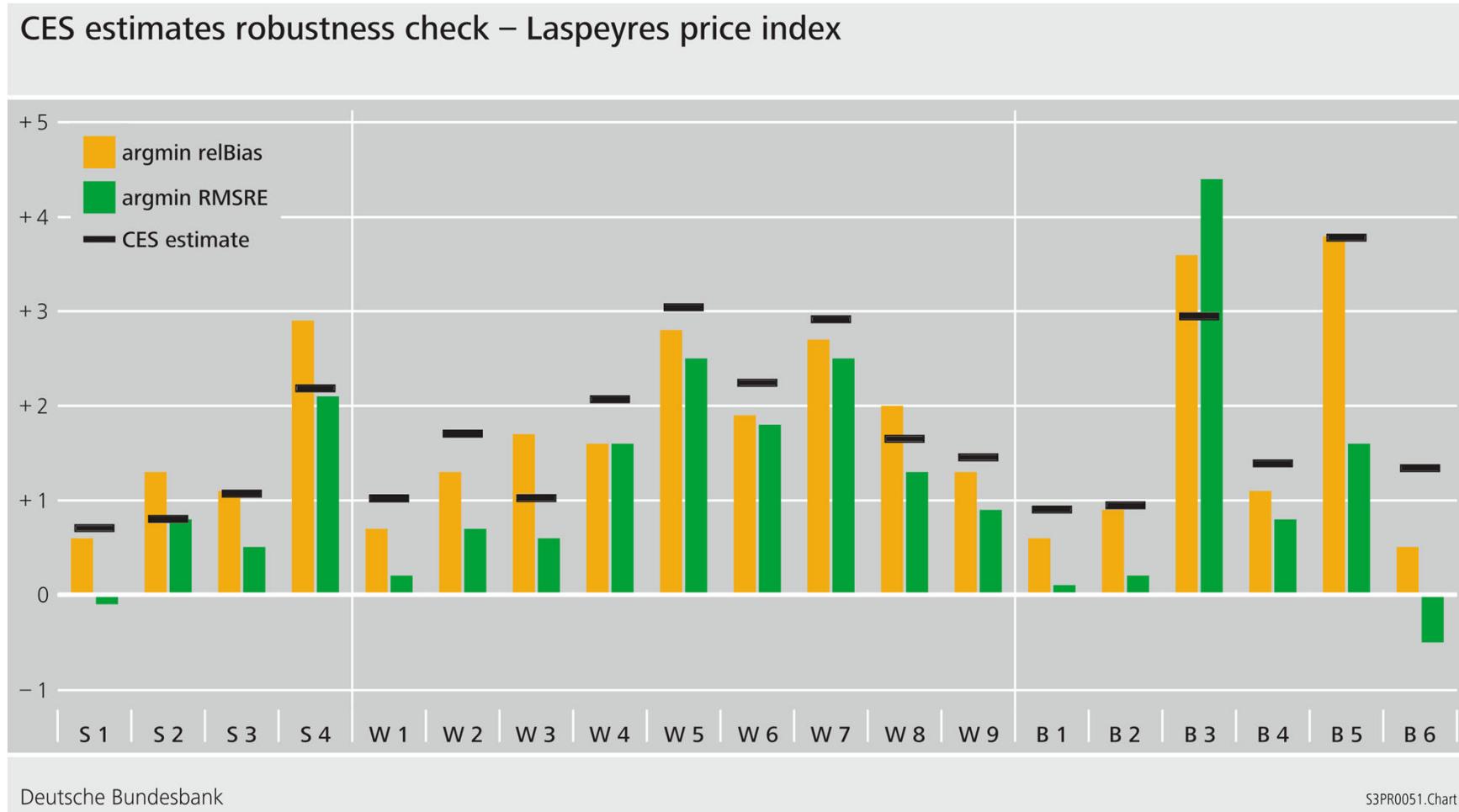
### Findings on substitution behaviour

- The median elasticity of substitution is 1.5, ranging from .7 to 3.8.
- All estimates are **statistically significantly greater than zero**; for 8 out of 19 sub-classes **the difference to iso-elasticity is insignificant**, while for the remaining 11 sub-classes substitution is found to even exceed unity elasticity.
- **In spirits, consumers are more willing to substitute between different types of whiskey (S 4) than is the case for brandy or vodka (S 1 and S 2).**
- **For both red and white wines, substitution is more pronounced for the New World (W 5 and W 7) than for European wines (W 4 and W 6).**
- **Also, the elasticity of substitution tends to be higher for 12 cans and 20 bottles of lager (B 3 and B 5), respectively, than for 4 packs (B 2 and B 4).**
- These results are **consistent with the findings of Elliott and O'Neill (2012).**
- Furthermore, comparing the CES regression results with the direct calculation of the generalised means, **the outcomes do not change qualitatively.**
- In particular, **the Carli index performs remarkably well** at the elementary level of a Laspeyres price index, questioning the argument of its “upward bias” – in fact, **it is the Jevons index that has a downward bias.**

## 5. Empirical results CES estimation results

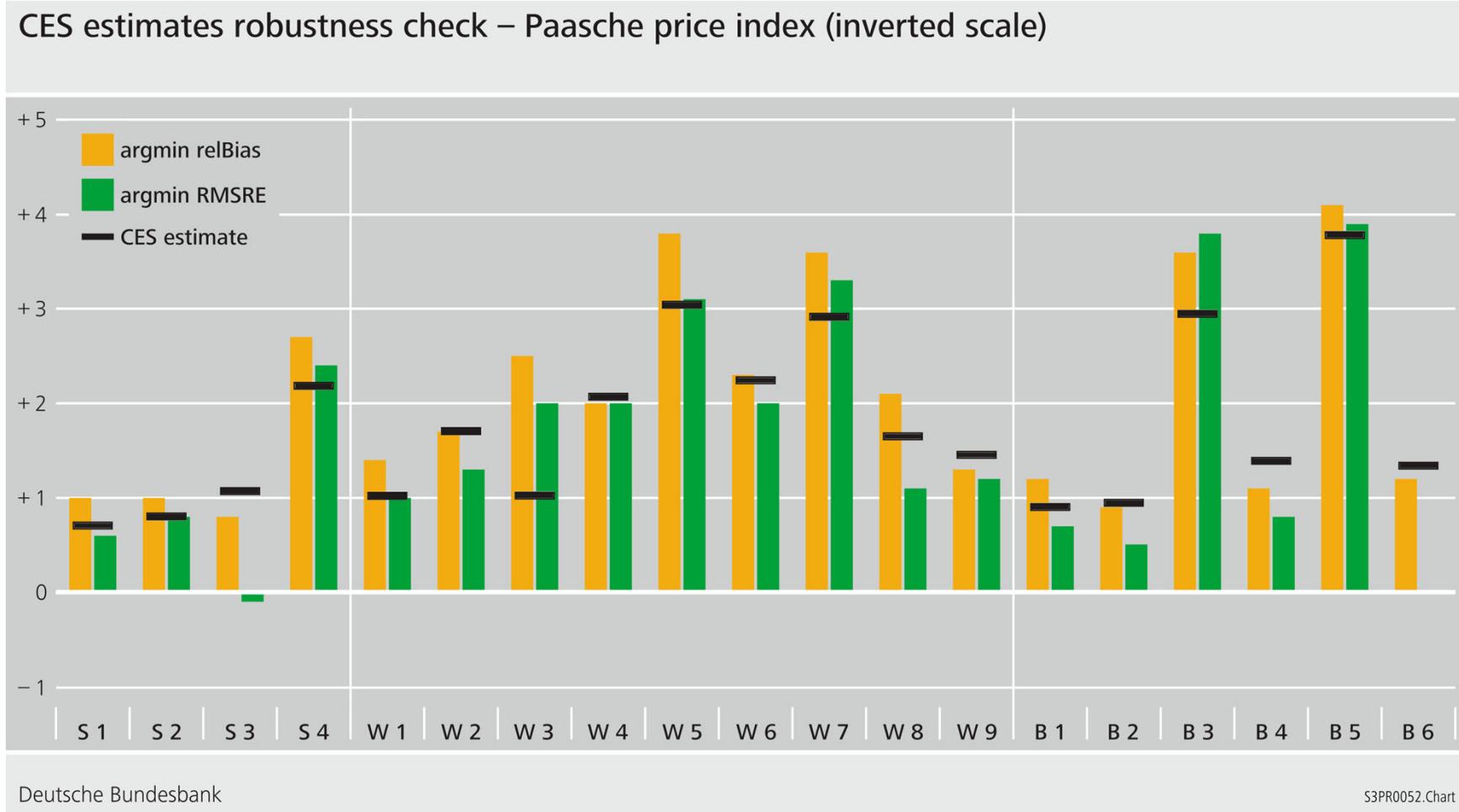


## 5. Empirical results Laspeyres price index: robustness

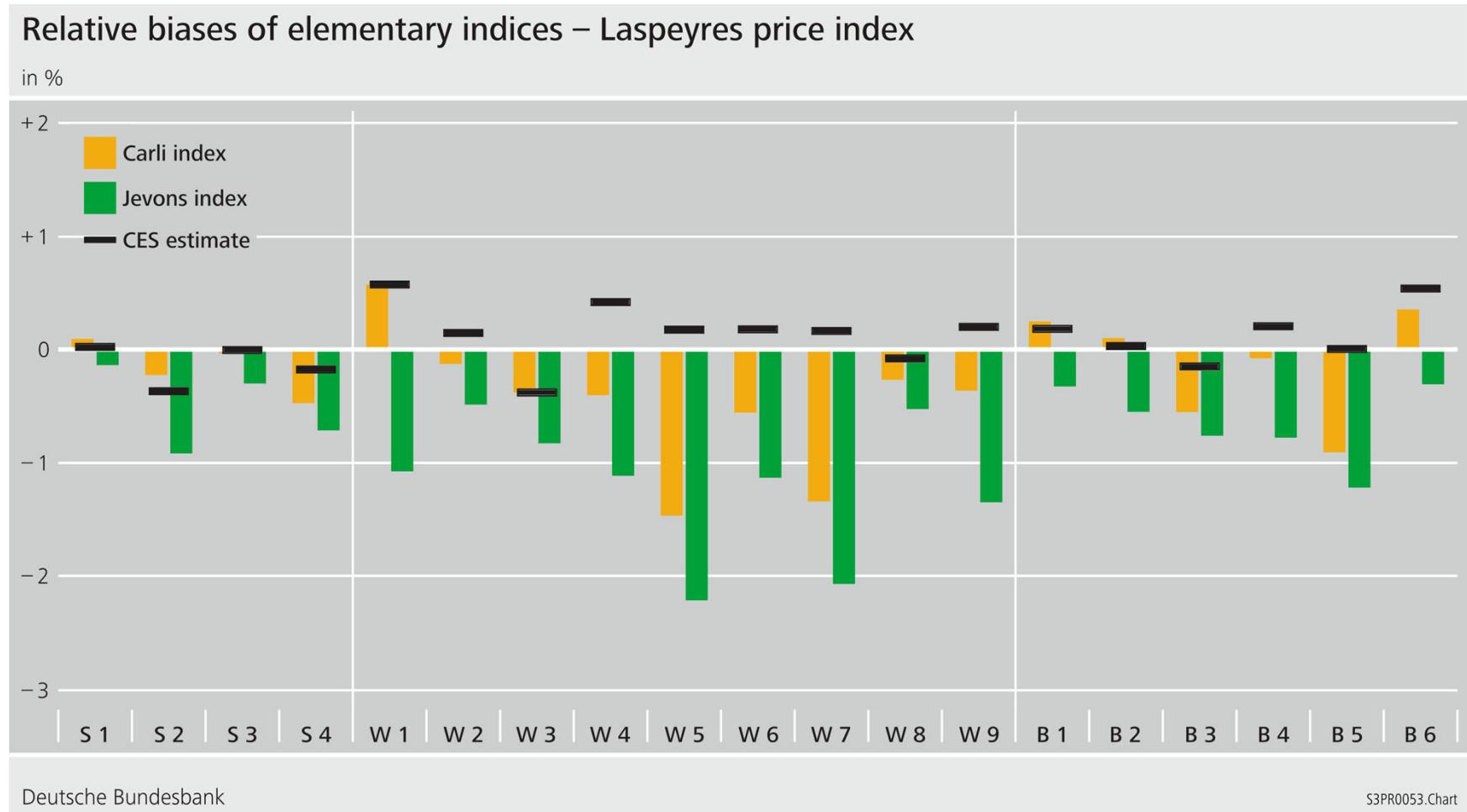


## 5. Empirical results

### Paasche price index: robustness

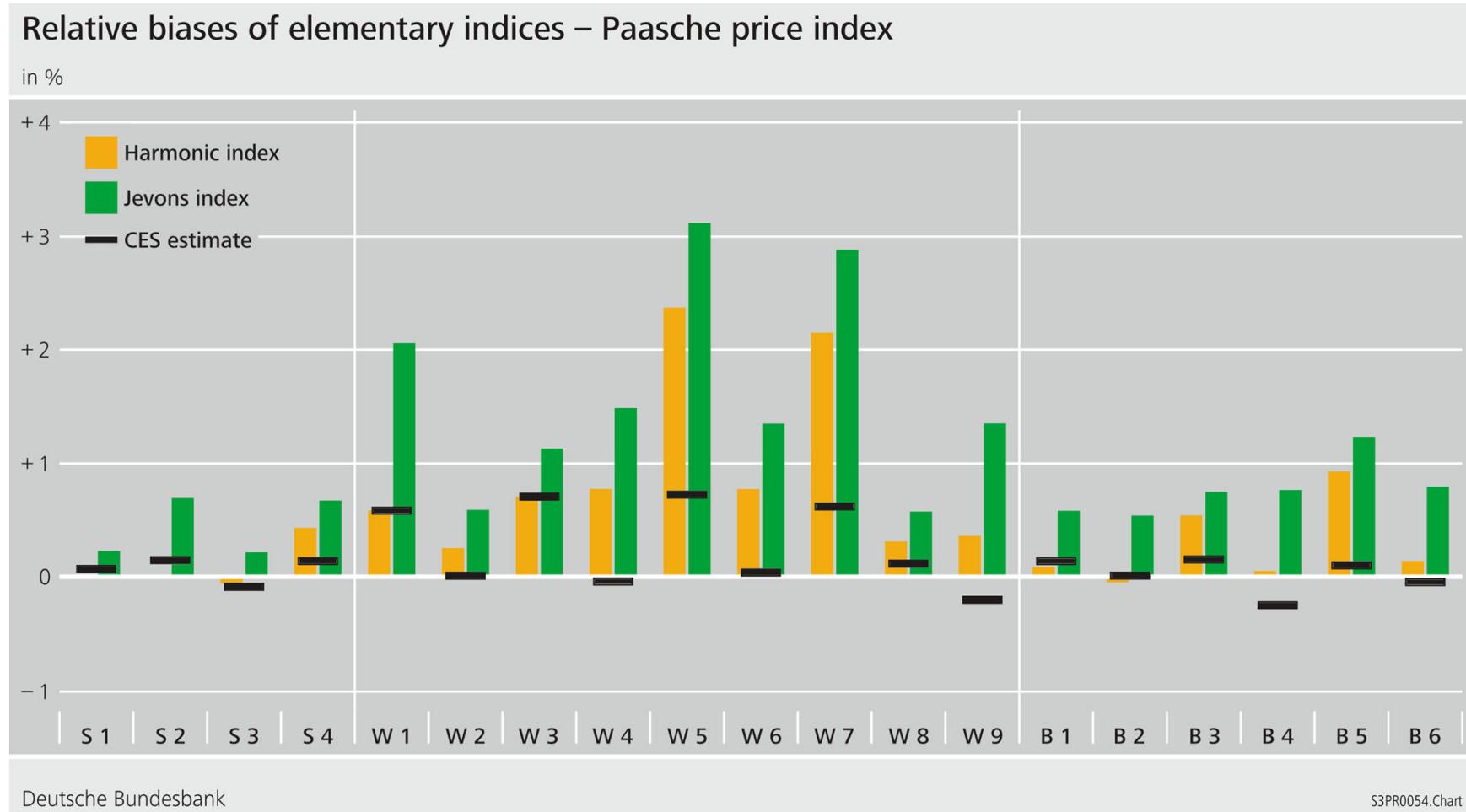


## 5. Empirical results Laspeyres price index: bias



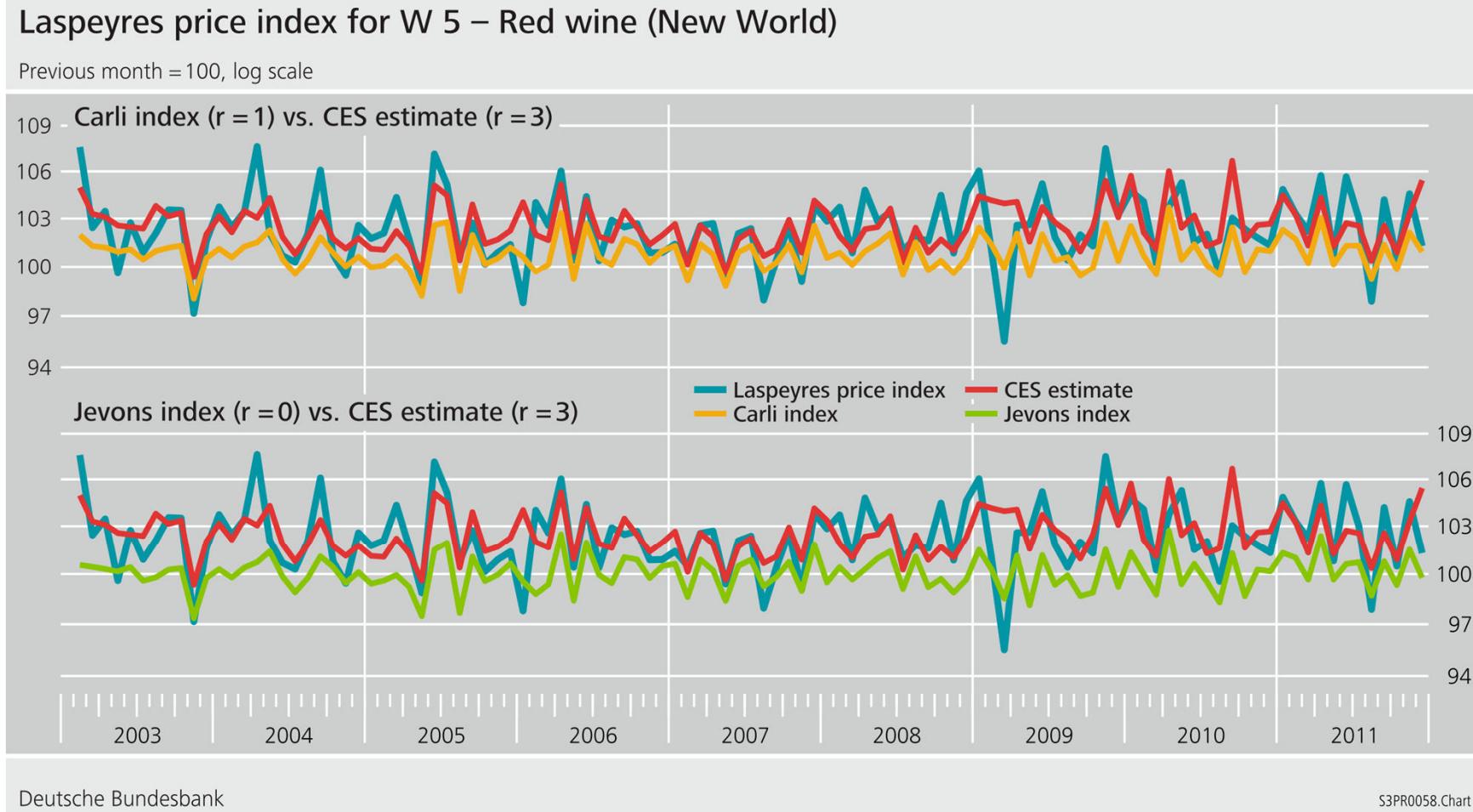
## 5. Empirical results

### Paasche price index: bias



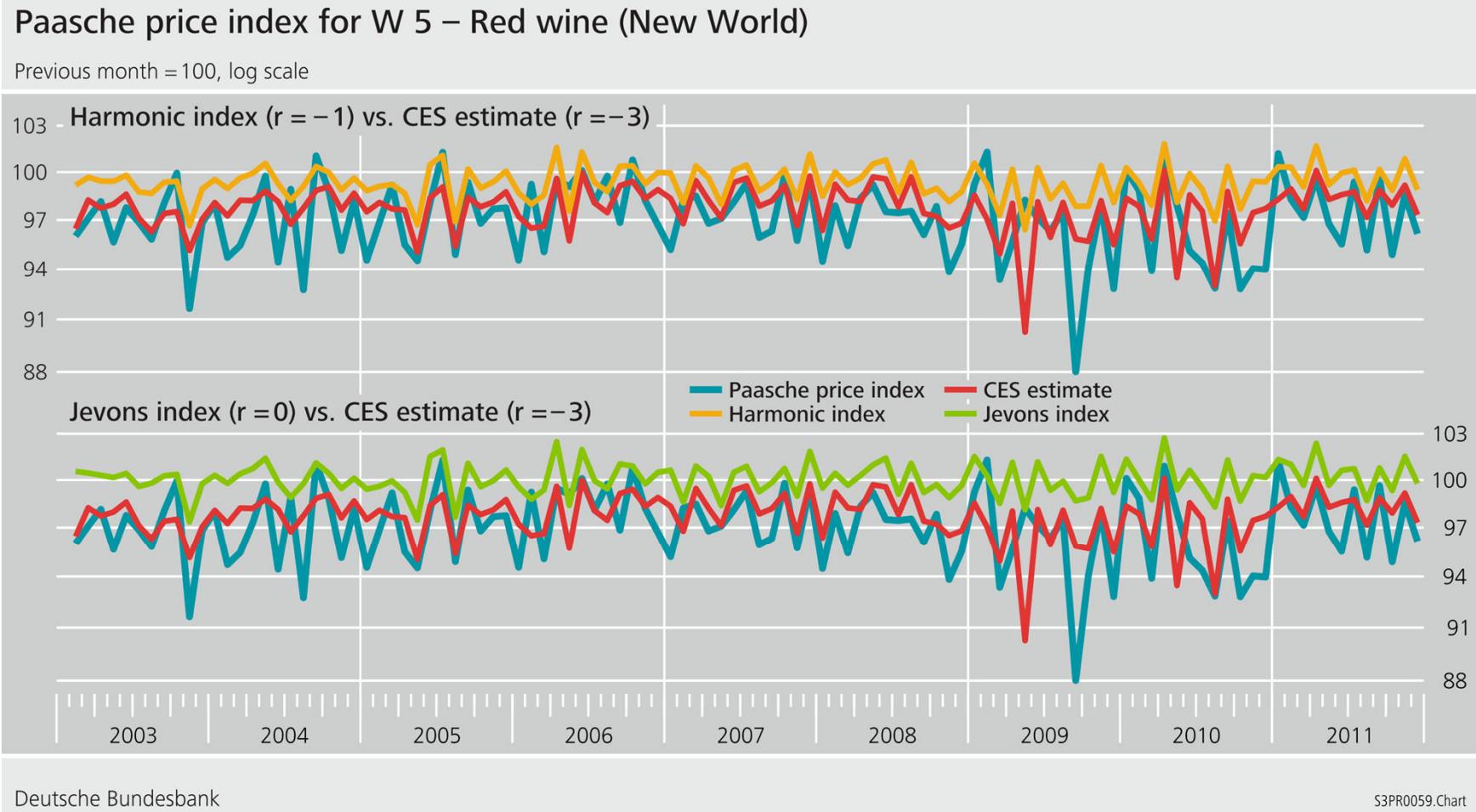
## 5. Empirical results

### Laspeyres price index: time series



## 5. Empirical results

### Paasche price index: time series



## 6. Discussion

- The problem of aggregational consistency demonstrates the need for **a weighting at the lowest possible level**.
- This would mean that, in the trade-off between **estimated weights/weights from secondary sources** on the one hand and the elementary bias of unweighted indices on the other, the balance would often tip **in favour of weighting**.
- **The biases at the elementary level** can, in some cases, reach such large dimensions that they become **relevant for the aggregate index**.
- There is a “**price**” **to be paid at the upper level** for suboptimal index formula selection at the lower level; thus, the need for **two-staged price indices to be accurately constructed** becomes obvious.
- **Disaggregation is a panacea!**
  
- Insofar as **no information on weights** is available, **studies on substitution** can help in **guiding the choice of the optimal elementary index** for a given measurement target.
- Often, even an expert judgement on substitutability **outperforms the test approach**.

## Annex

- What is the relation between “consistency in aggregation” on the one hand and **“constant elasticity of substitution”** on the other **in a Laspeyres-type price index**, such as the European Harmonised Index of Consumer Prices, **where a fixed set of weights is used at the aggregate level?**
- To reiterate, **the target index is given to be the Laspeyres price index.**
- **In this case, the CES utility function is needed solely to approximate the distribution of the period 0 expenditures among the commodities *within each of the elementary aggregates*.**
- That is, **all what is needed are relative price levels** rather than their changes.
- **Hence, the only issue here is the structure of the quantities purchased in the base period instead of substitution between items or strata over time!**