Bayesian functional models for population forecasting

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CPC centre for population change

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3. Forecast age-specific male and female population together when possible
4. As a generalisation of Lee-Carter model, functional data analysis provides a flexible approach
5. Bayesian model averaging between the independent and coherent functional models

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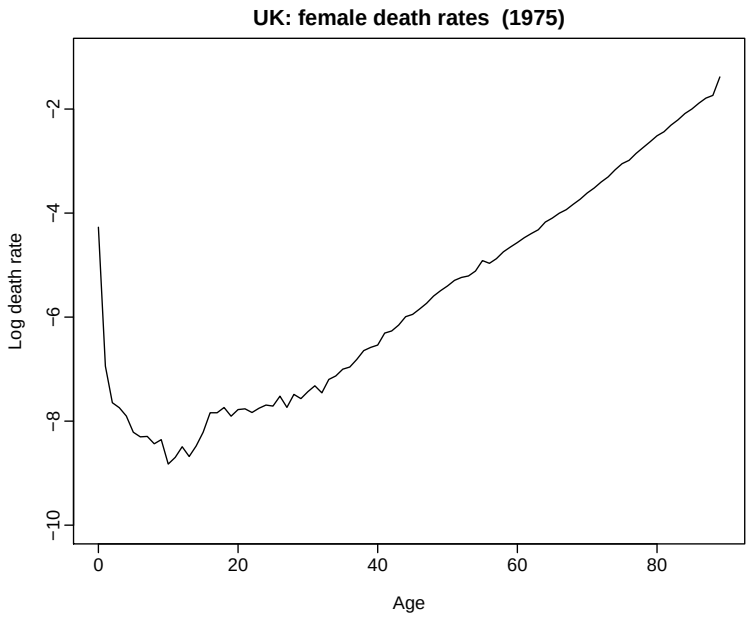
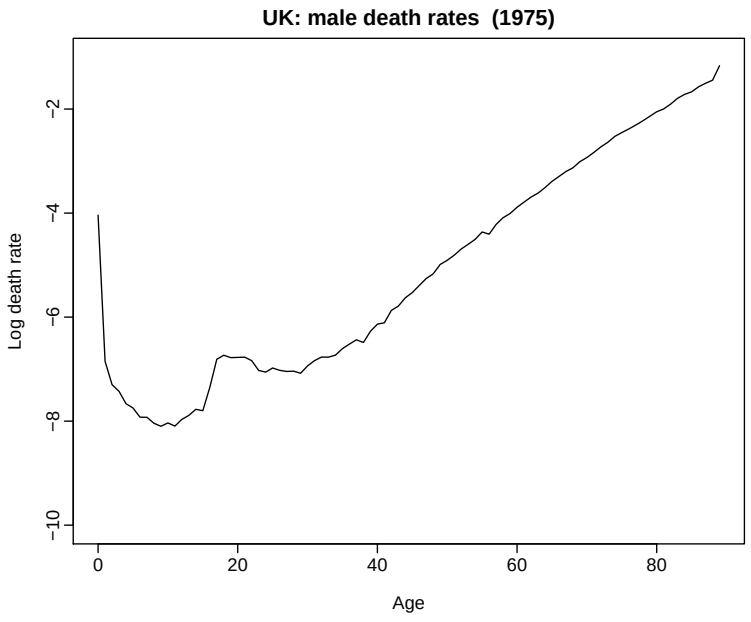
4. Aim to forecast population by age and gender from 2010 to 2030

Graphical display of mortality rates

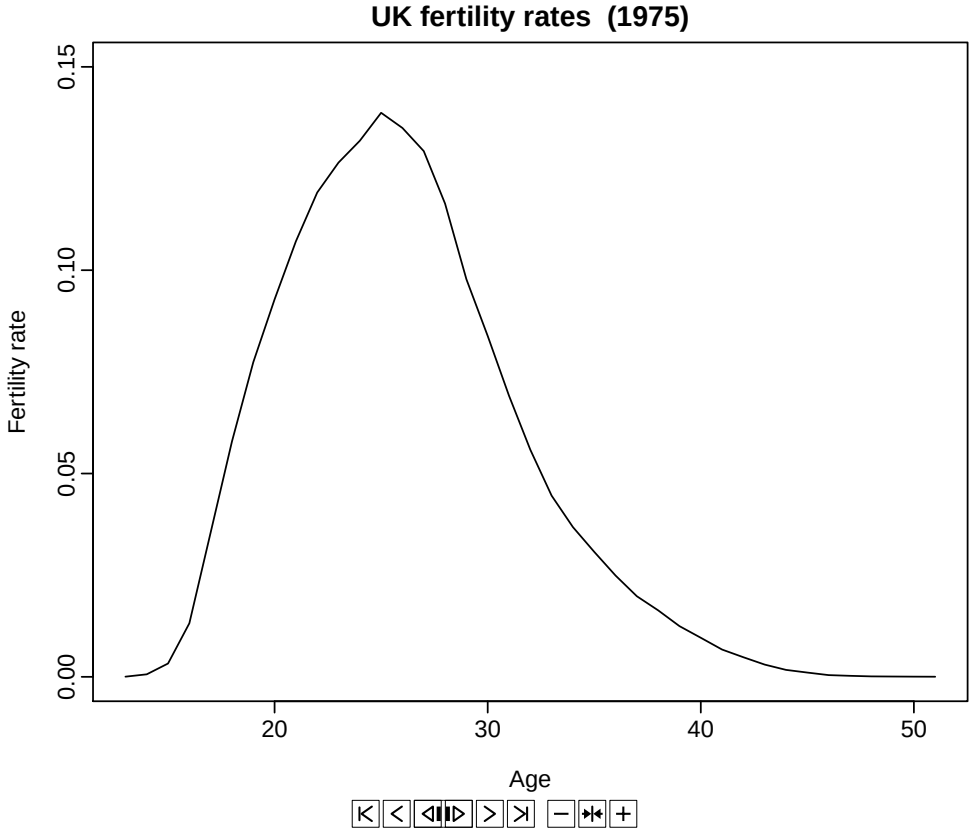
1. Mortality rates for single year of age (0–90+) between 1975 and 2009
2. Mortality rates dip in early childhood, climb in the teen years, stablise in the early 20s, and then steadily increase with age





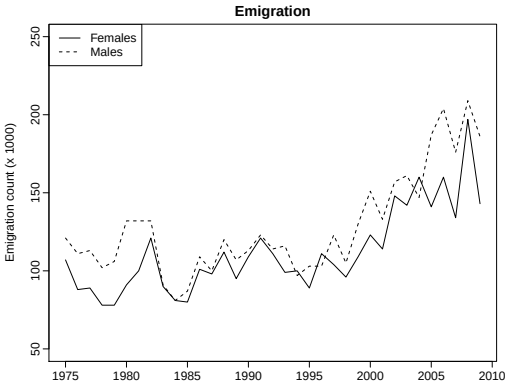
Graphical display of fertility rates

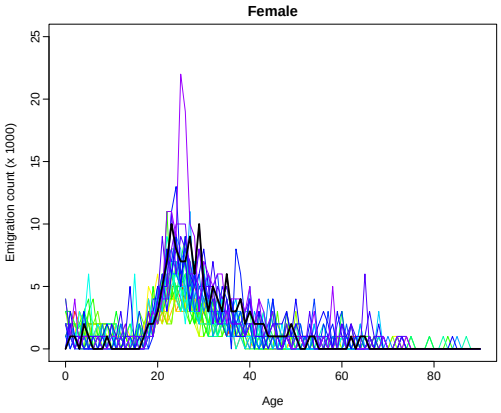
1. Age-specific fertility rates for ages (13–51) between 1975 and 2009
2. Increase in fertility rates at higher ages in more recent years caused by a tendency to postpone child-bearing

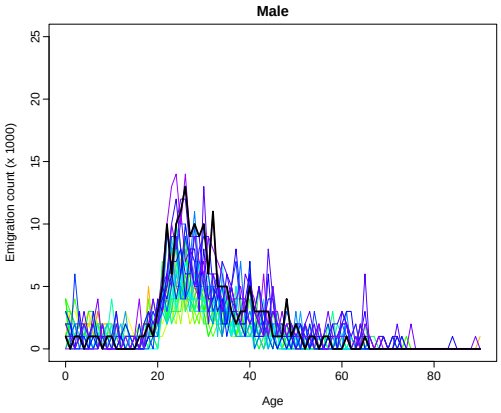


Graphical display of emigration rates

1. A slight increase over time in the emigration, but the patterns seem to be more volatile
2. Emigration data come from the International Passenger Survey

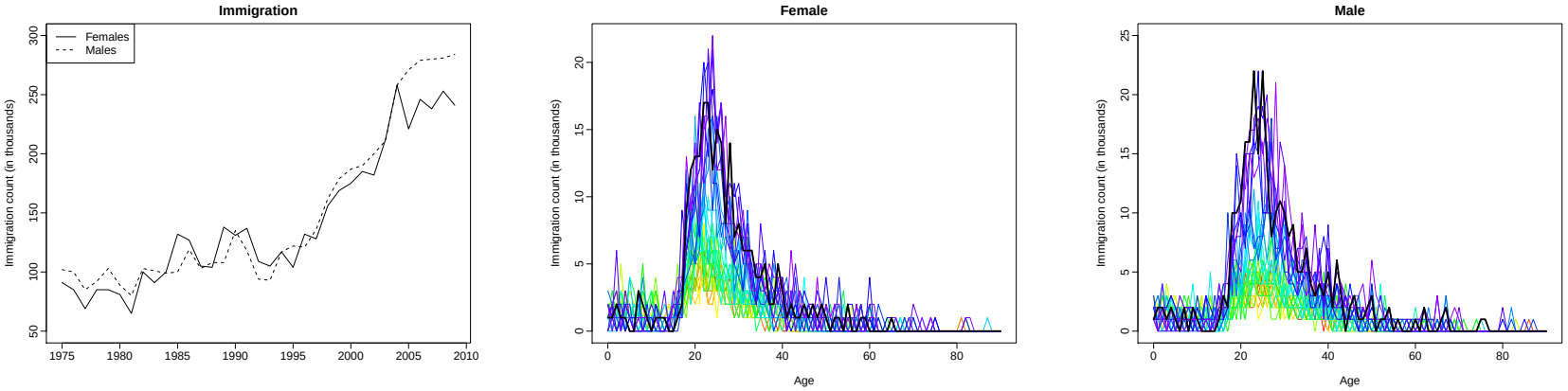






Graphical display of immigration counts

Immigration counts have been rapidly increasing from 1975 to 2009



Cohort component projection model

$$\begin{bmatrix} P^{\mathsf{F}}_{1,t+1} \\ P^{\mathsf{F}}_{2,t+1} \\ \vdots \\ P^{\mathsf{F}}_{z-1,t+1} \\ P^{\mathsf{F}}_{z,t+1} \end{bmatrix} = \begin{bmatrix} 0 & \dots & b^{\mathsf{F}}_{13,t} & \dots & b^{\mathsf{F}}_{51,t} & \dots & 0 \\ s^{\mathsf{F}}_{0,t} & 0 & 0 & & \dots & & 0 \\ 0 & s^{\mathsf{F}}_{1,t} & 0 & & \dots & & 0 \\ \vdots & & \ddots & & & & \vdots \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & \dots & & s^{\mathsf{F}}_{z-2,t} & 0 & 0 \\ 0 & 0 & \dots & & 0 & s^{\mathsf{F}}_{z-1,t} & s^{\mathsf{F}}_{z,t} \end{bmatrix} \times \begin{bmatrix} P^{\mathsf{F}}_{1,t} \\ P^{\mathsf{F}}_{2,t} \\ \vdots \\ P^{\mathsf{F}}_{z-1,t} \\ P^{\mathsf{F}}_{z,t} \end{bmatrix} + \boldsymbol{I}^{\mathsf{F}}_t,$$

- Growth matrix includes survival rates $s_{i,t},\ i = 0, \dots, 90+$ based on mortality and emigration, and fertility rates $b_{j,t},\ j = 13, \dots, 51$

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- Growth matrix includes survival rates $s_{i,t},\ i = 0, \dots, 90+$ based on mortality and emigration, and fertility rates $b_{j,t},\ j = 13, \dots, 51$
- $\boldsymbol{I}^{\mathsf{F}}_t = \left[I^{\mathsf{F}}_{0,t}, \dots, I^{\mathsf{F}}_{+90,t}\right]^\top$ denotes female immigration counts

Functional data analysis

1. Received increasing amount of attention (Hyndman and Ullah, 2007; Lazar and Denuit, 2009; Yang et al., 2010; Dowd et al., 2010; Cairns et al., 2011; D'Amato et al., 2011)

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2. Let $m_{t,i}$ represent the original data for ages $i = 0, \ldots 90+$ in year t. The Box-Cox transformation is applied to each component of population to alleviate heteroscedasticity and non-normality

$$g_{t,i} = \begin{cases} \frac{1}{\lambda}(m_{t,i}^{\lambda} - 1) & \text{if } 0 < \lambda \leq 1 \\ \ln(m_{t,i}) & \text{if } \lambda = 0 \end{cases}$$

where λ is the Box-Cox transformation parameter

Smoothing step

3. We apply a smoothing technique to transform the set of discrete data points, $g_{t,i}$, to continuous function

$$g_t(x_i) = \tau_t(x_i) + \sigma(x_i)\epsilon_t, \qquad t = 1, \ldots, n,$$

where $\sigma(x_i)$ models the variability for each age x_i, and $\epsilon_t \overset{iid}{\sim} N(0,1)$

Independent decomposition step

To model female and male data independently,

$$\tau_t^{\mathsf{F}}(x) = \mu^{\mathsf{F}}(x) + \sum_{k=1}^{K} \beta_{t,k}^{\mathsf{F}} \phi_k^{\mathsf{F}}(x) + e_t^{\mathsf{F}}(x), \qquad x \in [0, 90+] \tag{1}$$

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- $K < n$ is the number of retained components, which explain at least 99% of the total variation in data

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- $\widehat{\beta}_{t,k}^{\mathsf{F}}$ and $\widehat{v}(x)$ are drawn from their respective posterior distribution, where the likelihood is the Box-Cox normal and the prior densities of the variance parameters are inverse gamma distributions

Coherent decomposition step

4. We model female and male data jointly via multilevel functional principal component regression

$$\tau_t^{\mathsf{F}}(x) = \underbrace{\mu(x) + w^{\mathsf{F}}(x)}_{\mu^{\mathsf{F}}(x)} + R_t(x) + U_t^{\mathsf{F}}(x) + \varepsilon_t^{\mathsf{F}}(x), \qquad (2)$$

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- $\varepsilon_t^{\mathsf{F}}(x)$ represents the error term with finite variance $\delta(x)$
- Similarly, equation (2) also holds for males

Coherent decomposition step

5. The terms in equation (2) can be estimated by

$$\widehat{\mu}(x) = \frac{1}{2n}\sum_{j=1}^{2}\sum_{t=1}^{n}\tau_t^{(j)}(x), \quad \widehat{w}^{\mathsf{F}}(x) = \widehat{\mu}^{\mathsf{F}}(x) - \widehat{\mu}(x),$$

$$\widehat{R}_t(x) \approx \sum_{k=1}^{K}\widehat{\beta}_{t,k}\widehat{\phi}_k(x), \quad \widehat{U}_t^{\mathsf{F}}(x) \approx \sum_{l=1}^{L}\widehat{\gamma}_{t,l}^{\mathsf{F}}\widehat{\psi}_l^{\mathsf{F}}(x),$$

where

1. $\left\{\widehat{\beta}_{t,1},\ldots,\widehat{\beta}_{t,K}\right\}$ and $\left\{\widehat{\gamma}_{t,1}^{\mathsf{F}},\ldots,\widehat{\gamma}_{t,L}^{\mathsf{F}}\right\}$ represent sample principal component scores of $\widehat{R}_t(x)$ and $\widehat{U}_t^{\mathsf{F}}(x)$

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2. $\mathbf{\Phi} = \left\{\widehat{\phi}_1(x),\ldots,\widehat{\phi}_K(x)\right\}$ and $\mathbf{\Psi} = \left\{\widehat{\psi}_1^{\mathsf{F}}(x),\ldots,\widehat{\psi}_L^{\mathsf{F}}(x)\right\}$ represent the sample principal components of $\widehat{R}_t(x)$ and $\widehat{U}_t^{\mathsf{F}}(x)$

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2. $\mathbf{\Phi} = \left\{\widehat{\phi}_1(x),\ldots,\widehat{\phi}_K(x)\right\}$ and $\mathbf{\Psi} = \left\{\widehat{\psi}_1^{\mathsf{F}}(x),\ldots,\widehat{\psi}_L^{\mathsf{F}}(x)\right\}$ represent the sample principal components of $\widehat{R}_t(x)$ and $\widehat{U}_t^{\mathsf{F}}(x)$
3. K and L are the number of retained components, which explain at least 95% and 90% of the total variations in data

Forecasting step of coherent decomposition

6. Conditional on the estimated principal components $\mathbf{\Phi}$ and $\mathbf{\Psi}$ and smooth functions $\mathbf{\Gamma}$, point forecasts of $g^{\mathsf{F}}_{n+h}(x)$ are

$$\begin{aligned}\widehat{g}^{\mathsf{F}}_{n+h|n}(x) &= \mathsf{E}\left[g^{\mathsf{F}}_{n+h}(x)\middle|\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Gamma}\right] \\ &= \widehat{\mu}(x) + \widehat{w}^{\mathsf{F}}(x) + \sum_{k=1}^{K}\widehat{\beta}_{n+h|n,k}\widehat{\phi}_k(x) + \sum_{l=1}^{L}\widehat{\gamma}^{\mathsf{F}}_{n+h|n,l}\widehat{\psi}^{\mathsf{F}}_l(x)\end{aligned}$$

where $\widehat{\beta}_{n+h|n,k}$ and $\widehat{\gamma}^{\mathsf{F}}_{n+h|n,l}$ are the forecasted principal component scores, obtained from a univariate time-series method, such as the ARIMA

Prediction interval

Prediction interval of $g^{\mathsf{F}}_{n+h|n}(x)$ can be obtained using parametric bootstrap samples, given by

$$\widehat{g}^{b,\mathsf{F}}_{n+h|n}(x) = \widehat{\mu}(x) + \widehat{w}^{\mathsf{F}}(x) + \sum_{k=1}^{K} \widehat{\beta}^{b}_{n+h|n,k}\widehat{\phi}_k(x) + \sum_{l=1}^{L} \widehat{\gamma}^{b,\mathsf{F}}_{n+h|n,l}\widehat{\psi}^{\mathsf{F}}_l(x) + \widehat{\varepsilon}^{b}_{n+h}(x) + \widehat{\sigma}(x)\widehat{\epsilon}^{b}_{n+h}, \qquad b = 1, \ldots, B,$$

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4. $\widehat{\epsilon}^{b}_{n+h}$ is simulated from $N(0, 1)$

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2. θ_1 and θ_2 be vector of parameters associated with each model
3. Denote Δ as the quantity of interest, such as a combined forecast of age-specific mortality, then its posterior distribution given data D is

$$\begin{aligned}\Pr(\Delta|D) &= \sum_{r=1}^{2} \Pr(\Delta|M_r, D)\Pr(M_r|D) \\ &= \sum_{r=1}^{2} \Pr(\Delta|M_r, D) \underbrace{\frac{\Pr(D|M_r)\Pr(M_r)}{\sum_{l=1}^{2} \Pr(D|M_l)\Pr(M_l)}}_{\text{weight}}\end{aligned}$$

Harmonic estimator of marginal likelihood

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2. The marginal likelihood introduced by Newton & Raftery (1994) can be written as

$$\Pr(D|M_r) \approx \left(\frac{1}{T} \sum_{t=1}^{T} \Pr\left(D|\theta_{M_r}^{(t)}, M_r\right)^{-1} \right)^{-1}$$

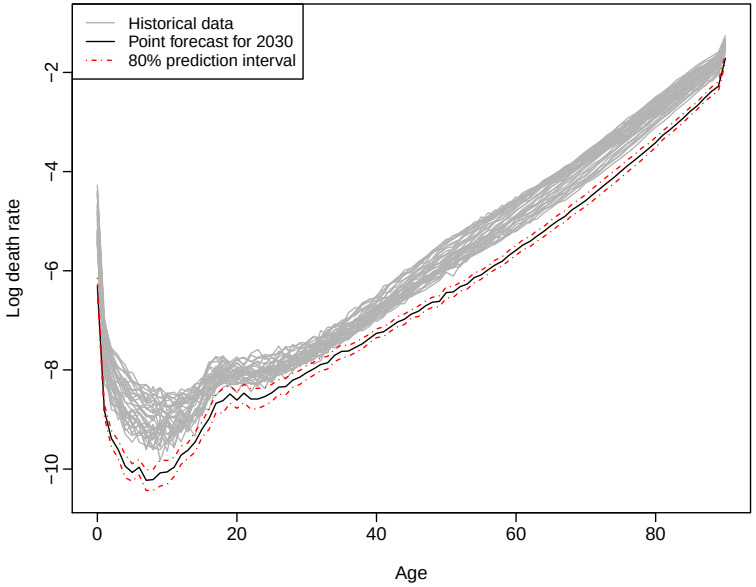
Harmonic estimator of marginal likelihood

1. Given equal prior probability for the two models, the posterior odds can be obtained by the ratios of two marginal likelihoods averaged over all Markov chain Monte Carlo iterations
2. The marginal likelihood introduced by Newton & Raftery (1994) can be written as

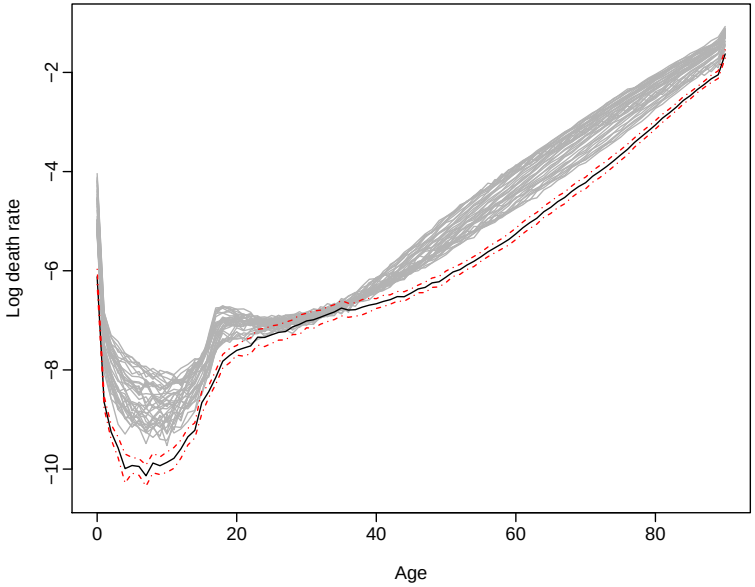
$$\Pr(D|M_r) \approx \left(\frac{1}{T} \sum_{t=1}^{T} \Pr\left(D|\theta_{M_r}^{(t)}, M_r\right)^{-1} \right)^{-1}$$

3. For mortality and emigration rates, coherent functional model has weight one. For immigration counts, independent functional model has weight one

Forecasts of mortality



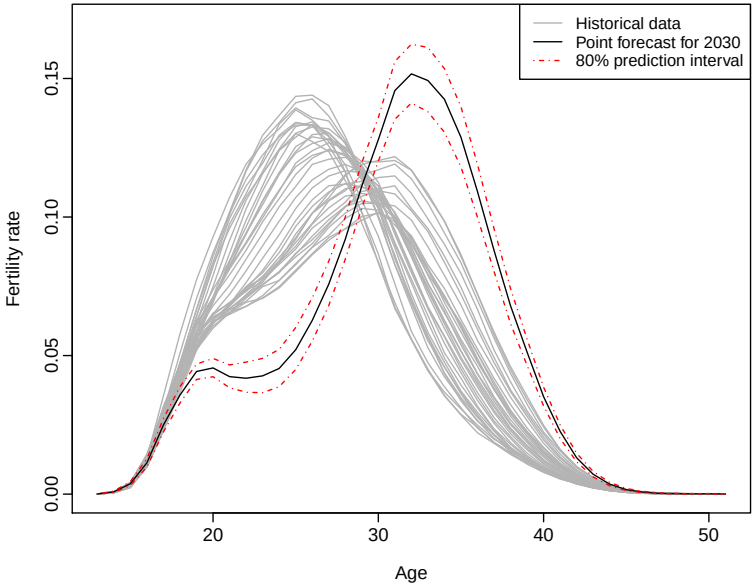
(a) Female

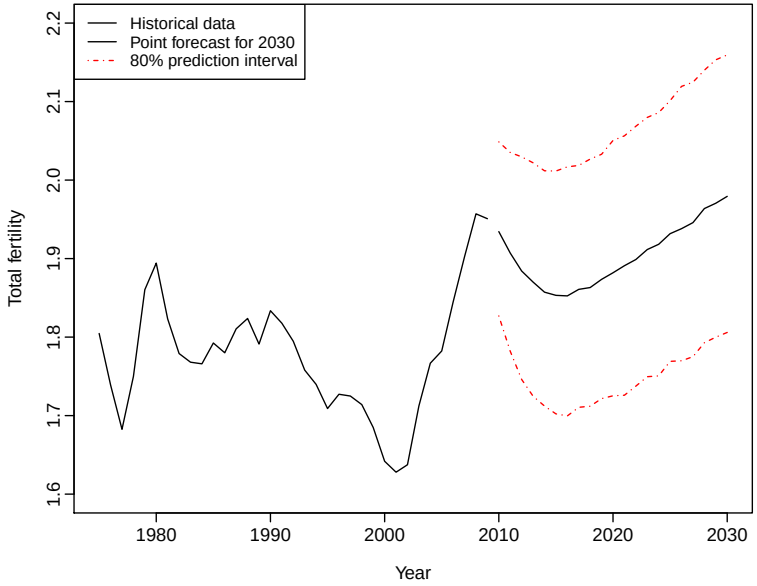


(b) Male

Log mortality rates are likely to drop in general, except for men aged around 30

Forecasts of fertility

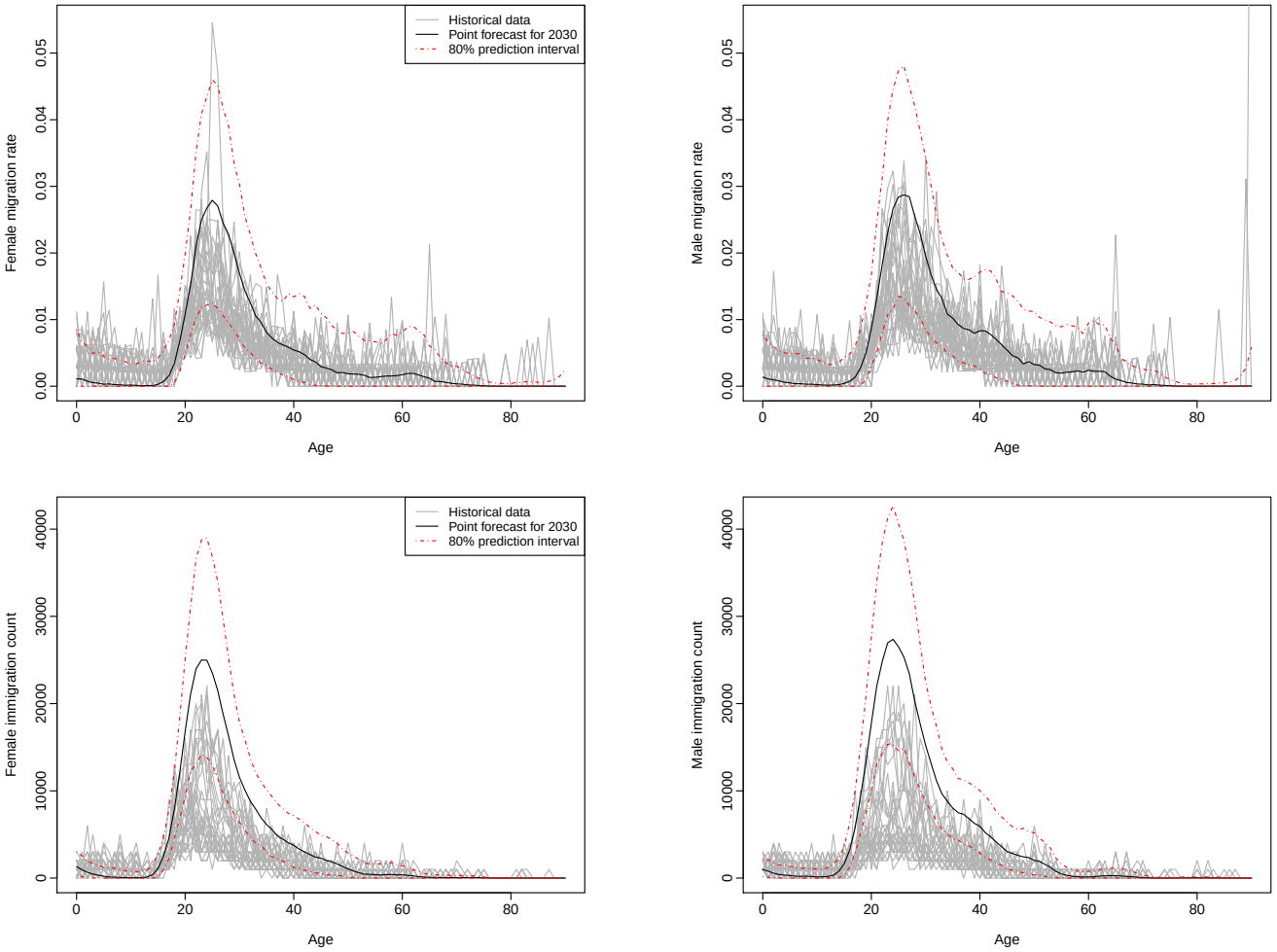




1) Total fertility rate will decrease until 2015, then increase thereafter

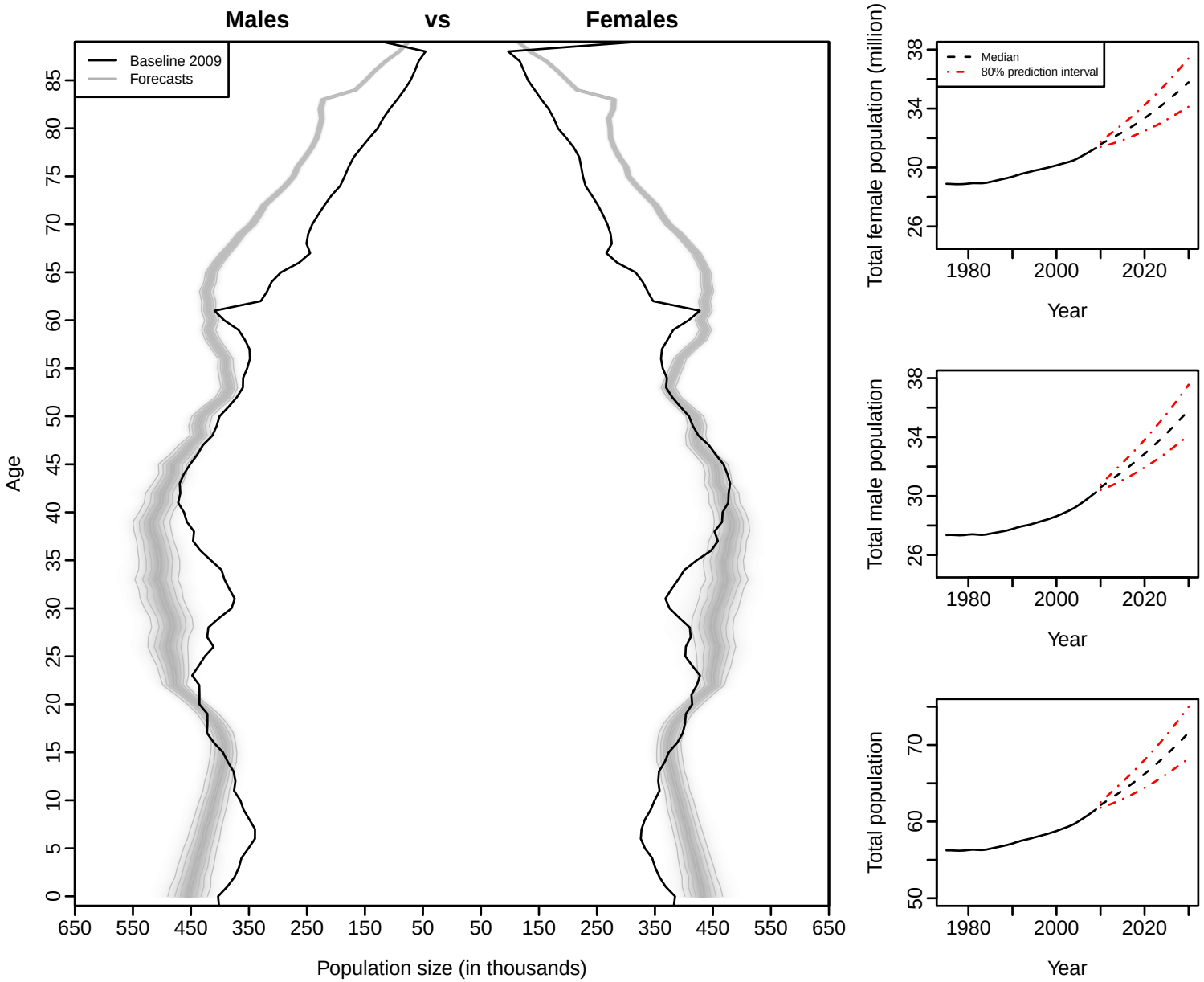
2) The slight declining, yet uncertain, fertility rates signal a possibility of another period of postponement

Forecasts of migration



A continuing increase in emigration rates and immigration counts for ages between 20 and 40

Forecasts of out-of-sample population in 2030



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5. Thank you

References:

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